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Quantum Giambelli formulas for isotropic Grassmannians

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Abstract Let X be a symplectic or odd orthogonal Grassmannian which parametrizes isotropic subspaces in a vector space equipped with a nondegenerate (skew) symmetric form. We prove quantum Giambelli formulas which express an arbitrary Schubert class in the small quantum cohomology ring of X as a polynomial in certain special Schubert classes, extending the authors' cohomological Giambelli formulas.

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0 Introduction

Let E be an even (respectively, odd) dimensional complex vector space equipped with a nondegenerate skew-symmetric (respectively, symmetric) bilinear form. Let X

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denote the Grassmannian which parametrizes the isotropic subspaces of E of some fixed dimension. The cohomology ring $H^*(X, \mathbb{Z})$ is generated by certain special Schubert classes, which for us are (up to a factor of two) the Chern classes of the universal quotient vector bundle over X . These special classes also generate the small quantum cohomology ring $QH(X)$, a q -deformation of $H^*(X, \mathbb{Z})$ whose structure constants are the three point, genus zero Gromov–Witten invariants of X . In [6], we proved a Giambelli formula in $H^*(X, \mathbb{Z})$, that is, a formula expressing a general Schubert class as an explicit polynomial in the special classes. Our goal in the present work is to extend this result to a formula that holds in $QH(X)$.

The quantum Giambelli formula for the usual type A Grassmannian was obtained by Bertram [2], and is in fact identical to the classical Giambelli formula. In the case of maximal isotropic Grassmannians, the corresponding questions were answered in [7, 8]. The main conclusions here are similar to those of loc. cit., provided that one uses the raising operator Giambelli formulas of [6] as the classical starting point. For an odd orthogonal Grassmannian, we prove that the quantum Giambelli formula is the same as the classical one. The result is more interesting when X is the Grassmannian $IG(n - k, 2n)$ parametrizing $(n - k)$ -dimensional isotropic subspaces of a symplectic vector space E of dimension $2n$. Our theorem in this case states that the quantum Giambelli formula for $IG(n - k, 2n)$ coincides with the classical Giambelli formula for $IG(n + 1 - k, 2n + 2)$, provided that the special Schubert class σ_{n+k+1} is replaced with $q/2$.

Although the two theorems in this article are analogous to those of [7, 8], their proofs are quite different. We prove the quantum Giambelli formula by using the quantum Pieri rule of [5], in a manner similar to [3] and [4, Remark 3]. However, unlike the previously known examples, for non-maximal isotropic Grassmannians no explicit recursion formula for the cohomological Giambelli polynomials is available, other than that given by the Pieri rule itself. We circumvent this difficulty by showing that a suitable recursion exists (Proposition 3). We also make essential use of a ring homomorphism from the stable cohomology ring of X to $QH(X)$ that is the identity on Schubert classes coming from $H^*(X, \mathbb{Z})$. The existence of this map (Propositions 4 and 5) may be of independent interest.

In a sequel to this paper, we will discuss the classical and quantum Giambelli formulas for even orthogonal Grassmannians.

1 Preliminary results

1.1 Classical Giambelli for IG

Choose $k \geq 0$ and consider the Grassmannian $IG = IG(n - k, 2n)$ of isotropic $(n - k)$ -dimensional subspaces of \mathbb{C}^{2n} , equipped with a symplectic form. A partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$ is k -strict if all of its parts greater than k are distinct integers. Following [5], the Schubert classes on IG are parametrized by the k -strict partitions whose diagrams fit in an $(n - k) \times (n + k)$ rectangle, i.e. $\lambda_1 \leq n + k$ and $\ell(\lambda) \leq n - k$; we denote the set of all such partitions by $\mathcal{P}(k, n)$. Given any partition $\lambda \in \mathcal{P}(k, n)$ and a complete flag of subspaces

$$F_{\bullet} : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n}$$

such that $F_{n+i} = F_{n-i}^{\perp}$ for $0 \leq i \leq n$, we have a Schubert variety

$$X_{\lambda}(F_{\bullet}) := \{\Sigma \in \text{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \ \forall 1 \leq j \leq \ell(\lambda)\},$$

where $\ell(\lambda)$ denotes the number of (non-zero) parts of λ and

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}.$$

This variety has codimension $|\lambda| = \sum \lambda_i$ and defines, via Poincaré duality, a Schubert class $\sigma_{\lambda} = [X_{\lambda}(F_{\bullet})]$ in $H^{2|\lambda|}(\text{IG}, \mathbb{Z})$. The Schubert classes σ_{λ} for $\lambda \in \mathcal{P}(k, n)$ form a free \mathbb{Z} -basis for the cohomology ring of IG. The *special Schubert classes* are defined by $\sigma_r = [X_r(F_{\bullet})] = c_r(\mathcal{Q})$ for $1 \leq r \leq n+k$, where \mathcal{Q} denotes the universal quotient bundle over IG.

The classical Giambelli formula for IG is expressed using Young's *raising operators* [11, p. 199]. We first agree that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r < 0$. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ with finite support and $i < j$, we set $R_{ij}(\alpha) = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_j - 1, \dots)$; a raising operator R is any monomial in these R_{ij} 's. Define $m_{\alpha} = \prod_i \sigma_{\alpha_i}$ and $Rm_{\alpha} = m_{R\alpha}$ for any raising operator R . For any k -strict partition λ , we consider the operator

$$R^{\lambda} = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j > 2k + j - i} (1 + R_{ij})^{-1}$$

where the first product is over all pairs $i < j$ and second product is over pairs $i < j$ such that $\lambda_i + \lambda_j > 2k + j - i$. The main result of [6] states that the *Giambelli formula*

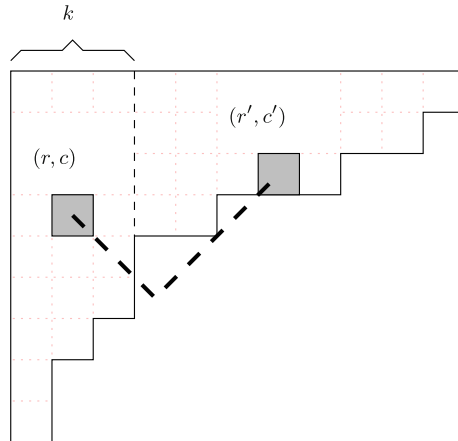
$$\sigma_{\lambda} = R^{\lambda} m_{\lambda} \tag{1}$$

holds in the cohomology ring of $\text{IG}(n-k, 2n)$.

1.2 Classical Pieri for IG

As is customary, we will represent a partition by its Young diagram of boxes; this is used to define the containment relation for partitions. Given two diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ (i.e., the set-theoretic difference $\nu \setminus \mu$) is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row).

We say that the box $[r, c]$ in row r and column c of a k -strict partition λ is *k-related* to the box $[r', c']$ if $|c - k - 1| + r = |c' - k - 1| + r'$. For instance, the grey boxes in the following partition are *k-related*.



For any two k -strict partitions λ and μ , we write $\lambda \rightarrow \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

- (1) if one of the first k columns of μ has the same number of boxes as the same column of λ , then the bottom box of this column is k -related to at most one box of $\mu \setminus \lambda$; and
- (2) if a column of μ has fewer boxes than the same column of λ , then the removed boxes and the bottom box of μ in this column must each be k -related to exactly one box of $\mu \setminus \lambda$, and these boxes of $\mu \setminus \lambda$ must all lie in the same row.

Let \mathbb{A} denote the set of boxes of $\mu \setminus \lambda$ in columns $k + 1$ through $k + n$ which are not mentioned in (1) or (2) above, and define $N(\lambda, \mu)$ to be the number of connected components of \mathbb{A} which do not have a box in column $k + 1$. Here two boxes are connected if they share at least a vertex. In [5, Thm. 1.1] we proved that the Pieri rule

$$\sigma_p \cdot \sigma_\lambda = \sum_{\substack{\lambda \rightarrow \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda, \mu)} \sigma_\mu \quad (2)$$

holds in $H^*(IG, \mathbb{Z})$, for any $p \in [1, n + k]$.

1.3 A recursion formula for IG

In the following sections we will work in the stable cohomology ring $\mathbb{H}(IG_k)$, which is the inverse limit in the category of graded rings of the system

$$\cdots \leftarrow H^*(IG(n - k, 2n), \mathbb{Z}) \leftarrow H^*(IG(n + 1 - k, 2n + 2), \mathbb{Z}) \leftarrow \cdots$$

The ring $\mathbb{H}(IG_k)$ has a free \mathbb{Z} -basis of Schubert classes σ_λ , one for each k -strict partition λ , and may be presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \sigma_2, \dots]$ modulo the relations

$$\sigma_r^2 + 2 \sum_{i=1}^r (-1)^i \sigma_{r+i} \sigma_{r-i} = 0 \quad \text{for } r > k. \quad (3)$$

There is a natural surjective ring homomorphism $\mathbb{H}(\text{IG}_k) \rightarrow \mathbb{H}(\text{IG}(n-k, 2n), \mathbb{Z})$ that maps σ_λ to σ_λ , when $\lambda \in \mathcal{P}(k, n)$, and to zero, otherwise. The Giambelli formula (1) and Pieri rule (2) are both valid in $\mathbb{H}(\text{IG}_k)$. We begin with some elementary consequences of these theorems.

For any k -strict partition λ of length ℓ , we define the sets of pairs

$$\begin{aligned} \mathcal{A}(\lambda) &= \{(i, j) \mid \lambda_i + \lambda_j \leq 2k + j - i \text{ and } 1 \leq i < j \leq \ell\} \\ \mathcal{C}(\lambda) &= \{(i, j) \mid \lambda_i + \lambda_j > 2k + j - i \text{ and } 1 \leq i < j \leq \ell\} \end{aligned}$$

and two integer vectors $a = (a_1, \dots, a_\ell)$ and $c = (c_1, \dots, c_\ell)$ by setting

$$a_i = \#\{j \mid (i, j) \in \mathcal{A}(\lambda)\}, \quad c_i = \#\{j \mid (i, j) \in \mathcal{C}(\lambda)\}$$

for each i .

Proposition 1 *We have $\lambda_i - c_i \geq \lambda_j - c_j$ for each $i < j \leq \ell$.*

Proof Observe that the desired inequality is equivalent to

$$\lambda_i - \lambda_j \geq \#\{r \leq \ell \mid (i, r) \in \mathcal{C}(\lambda)\} - \#\{r \leq \ell \mid (j, r) \in \mathcal{C}(\lambda)\}. \quad (4)$$

Let $j = i + r$ and let s (respectively t) be maximal such that $(i, s) \in \mathcal{C}(\lambda)$ (respectively, $(j, t) \in \mathcal{C}(\lambda)$). Assume first that t exists, hence s exists and $s \geq t$. The inequality (4) then becomes $\lambda_i - \lambda_{i+r} \geq s - t + r$. If $t = s$, this is true because $(j, j+1) \in \mathcal{C}(\lambda)$ and λ is k -strict, hence $\lambda_i > \lambda_{i+1} > \dots > \lambda_{i+r}$. Otherwise we have $t < s \leq \ell$, $\lambda_i + \lambda_s \geq 2k + 1 + s - i$, and $\lambda_{i+r} + \lambda_{t+1} \leq 2k + t + 1 - i - r$. It follows that $\lambda_i - \lambda_{i+r} \geq s - t + r + (\lambda_{t+1} - \lambda_s) \geq s - t + r$.

Next we assume that t does not exist, so that either $j = \ell$ or the pair $(j, j+1)$ lies in $\mathcal{A}(\lambda)$ and

$$\lambda_j + \lambda_{j+1} \leq 2k + 1. \quad (5)$$

If s does not exist, the inequality is obvious. Otherwise, we must show that $\lambda_i - \lambda_j \geq s - i$, knowing that $(i, s) \in \mathcal{C}(\lambda)$, that is,

$$\lambda_i + \lambda_s \geq 2k + 1 + s - i. \quad (6)$$

Suppose first that $\lambda_s \geq \lambda_j$. If $\lambda_s > k$ then we have $\lambda_i > \lambda_{i+1} > \dots > \lambda_s$ and hence $\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq s - i$. Otherwise $\lambda_s \leq k$ and (6) gives

$$\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq \lambda_i - k \geq s - i + 1 + (k - \lambda_s) \geq s - i.$$

Finally, suppose that $\lambda_s < \lambda_j$, so in particular $j + 1 \leq s$. Then (5) and (6) give

$$\begin{aligned}\lambda_i - \lambda_j &\geq \lambda_i + (\lambda_{j+1} - 2k - 1) \geq (2k + 1 + s - i - \lambda_s) + \lambda_{j+1} - 2k - 1 \\ &= (\lambda_{j+1} - \lambda_s) + (s - i) \geq s - i.\end{aligned}$$

□

Proposition 1 implies that for any λ , the composition $\lambda - c$ is a partition, while $\lambda + a$ is a strict partition.

Proposition 2 *For any k -strict partition λ , the Giambelli polynomial $R^\lambda m_\lambda$ for σ_λ involves only generators σ_p with $p \leq \lambda_1 + a_1 + \lambda_2 + a_2$.*

Proof We have

$$R^\lambda m_\lambda = \prod_{1 \leq i < j \leq \ell} \frac{1 - R_{ij}}{1 + R_{ij}} \prod_{(i,j) \in \mathcal{A}(\lambda)} (1 + R_{ij}) m_\lambda = \sum_{v \in N} \prod_{1 \leq i < j \leq \ell} \frac{1 - R_{ij}}{1 + R_{ij}} m_v$$

where N is the multiset of integer vectors defined by

$$N = \left\{ \prod_{(i,j) \in S} R_{ij} \lambda \mid S \subset \mathcal{A}(\lambda) \right\}.$$

If $m > 0$ is the least integer such that $2m \geq \ell$, then we have

$$\prod_{1 \leq i < j \leq 2m} \frac{1 - R_{ij}}{1 + R_{ij}} = \text{Pfaffian} \left(\frac{1 - R_{ij}}{1 + R_{ij}} \right)_{1 \leq i < j \leq 2m}. \quad (7)$$

Equation (7) follows from Schur's classical identity [9, Sect. IX]

$$\prod_{1 \leq i < j \leq 2m} \frac{x_i - x_j}{x_i + x_j} = \text{Pfaffian} \left(\frac{x_i - x_j}{x_i + x_j} \right)_{1 \leq i, j \leq 2m}.$$

Note that each single entry in the Pfaffian (7) expands according to the formula

$$\frac{1 - R_{12}}{1 + R_{12}} m_{c,d} = \sigma_c \sigma_d - 2 \sigma_{c+1} \sigma_{d-1} + 2 \sigma_{c+2} \sigma_{d-2} - \cdots + (-1)^d 2 \sigma_{c+d}.$$

By Proposition 1, we know that $\lambda + a = (\lambda_1 + a_1, \lambda_2 + a_2, \dots, \lambda_\ell + a_\ell)$ is a strict partition, hence $\lambda_i + a_i + \lambda_j + a_j \leq \lambda_1 + a_1 + \lambda_2 + a_2$ for any distinct i and j . Since we furthermore have $v_i \leq \lambda_i + a_i$, for any $v \in N$, the result follows. □

Corollary 1 *For any $\lambda \in \mathcal{P}(k, n)$ the stable Giambelli polynomial for σ_λ involves only special classes σ_p with $p \leq 2n + 2k - 1$.*

Given any partition λ , we set $\lambda^* = (\lambda_2, \lambda_3, \dots, \lambda_\ell)$.

Lemma 1 *Let λ and ν be k -strict partitions such that $\nu_1 > \max(\lambda_1, \ell(\lambda) + 2k)$ and let $p, m \geq 0$. Then the coefficient of σ_ν in the Pieri product $\sigma_p \cdot \sigma_\lambda$ is equal to the coefficient of $\sigma_{(\nu_1+m, \nu^*)}$ in the product $\sigma_{p+m} \cdot \sigma_\lambda$.*

Proof Since the box $[\ell(\lambda), 1]$ is k -related to $[1, \ell(\lambda) + 2k]$ and $\nu_1 > \ell(\lambda) + 2k$, it follows that $\lambda \rightarrow \nu$ if and only if $\lambda \rightarrow (\nu_1 + m, \nu^*)$. In this case all of the boxes $[1, c]$ for $\max(\lambda_1, \ell(\lambda) + 2k) < c \leq \nu_1$ are contained in the rightmost component of the subset \mathbb{A} of $\nu \setminus \lambda$ defined in Sect. 1.2. Since replacing ν with $(\nu_1 + m, \nu^*)$ simply adds m boxes to this component, we deduce that $N(\lambda, \nu) = N(\lambda, (\nu_1 + m, \nu^*))$. \square

Proposition 3 *Let λ be a k -strict partition. Then there exist unique coefficients $a_{p,\mu} \in \mathbb{Z}$ for $p \geq \lambda_1$ and (p, μ) a k -strict partition, such that the recursive identity*

$$\sigma_\lambda = \sum_{p \geq \lambda_1} \sum_{\mu : (p, \mu) \text{ } k\text{-strict}} a_{p,\mu} \sigma_p \sigma_\mu \quad (8)$$

holds in $\mathbb{H}(\text{IG}_k)$. Furthermore, $a_{p,\mu} = 0$ whenever $\mu \not\subset \lambda^$, or when $\lambda \in \mathcal{P}(k, n)$ and $p \geq 2n + 2k$.*

Proof The Pieri rule (2) implies that

$$\sigma_\lambda = \sigma_{\lambda_1} \sigma_{\lambda^*} - \sum_{\substack{\lambda^* \rightarrow \nu \neq \lambda \\ |\nu| = |\lambda|}} 2^{N(\lambda^*, \nu)} \sigma_\nu.$$

Since all partitions ν in the sum satisfy $\nu_1 > \lambda_1$ and $\nu^* \subset \lambda^*$, the existence of the coefficients $a_{p,\mu}$ follows by descending induction on λ_1 , and they satisfy $a_{(p,\mu)} = 0$ for $\mu \not\subset \lambda^*$. The uniqueness is true because the set of all products $\sigma_p \cdot \sigma_\mu$ for which (p, μ) is a k -strict partition is linearly independent in $\mathbb{H}(\text{IG}_k)$. In fact, if the Schubert classes of $\mathbb{H}(\text{IG}_k)$ are ordered by the dominance order of partitions, then the lowest term of the product $\sigma_p \cdot \sigma_\mu$ is the class $\sigma_{(p,\mu)}$.

On the other hand, Proposition 2 implies that there are coefficients $b_{p,\mu}$, indexed by integers $p \in [\lambda_1, \lambda_1 + a_1 + \lambda_2 + a_2]$ and k -strict partitions μ , such that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{\lambda_1+a_1+\lambda_2+a_2} \sum_{|\mu|=|\lambda|-p} b_{p,\mu} \sigma_p \sigma_\mu.$$

In fact, if m_ν is any monomial appearing in the stable Giambelli formula $\sigma_\lambda = R^\lambda m_\lambda$, then $\lambda_1 \leq \max_i(v_i) \leq \lambda_1 + a_1 + \lambda_2 + a_2$. If $\lambda_1 > |\lambda^*|$, then the uniqueness of the coefficients $a_{p,\mu}$ implies that $b_{p,\mu} = a_{p,\mu}$. In particular, we have $a_{p,\mu} = 0$ for $p > \lambda_1 + a_1 + \lambda_2 + a_2$ in this case.

Now let $\lambda \in \mathcal{P}(k, n)$. Choose $m > |\lambda^*|$ and set $\lambda' = (\lambda_1 + m, \lambda^*)$. By the above discussion, there are coefficients $c_{p,\mu} \in \mathbb{Z}$ such that

$$\sigma_{\lambda'} = \sum_{p=\lambda_1+m}^{2n+2k-1+m} \sum_{\mu \subset \lambda^*} c_{p,\mu} \sigma_p \sigma_\mu. \quad (9)$$

We claim that the difference

$$\sigma_\lambda - \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu \subset \lambda^*} c_{p+m,\mu} \sigma_p \sigma_\mu \quad (10)$$

is a linear combination of classes σ_v for partitions $v \in \mathcal{P}(k, n)$ with $v_1 > \lambda_1$. To see this, notice that we must have $c_{\lambda_1+m,\lambda^*} = 1$, and hence the coefficient of σ_λ in the sum is equal to one. It follows that the difference (10) is equal to a linear combination of classes σ_v for which $v_1 > \lambda_1$. Furthermore, if $v_1 > n + k$, then Lemma 1 implies that the coefficient of σ_v in the sum in (10) is equal to the coefficient of $\sigma_{(v_1+m,v^*)}$ on the right hand side of (9), which is zero. This proves the claim. Finally, the proposition follows from the claim by descending induction on λ_1 . \square

Remark 1 One can be more precise about the recursion formula (8) in the case when the k -strict partition λ satisfies $\lambda_1 > \ell(\lambda) + 2k$. If the Pieri rule reads

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda^*} = \sum_{p \geq \lambda_1} \sum_{\mu \subset \lambda^*} 2^{n(p,\mu)} \sigma_{p,\mu}$$

then we have

$$\sigma_\lambda = \sum_{p \geq \lambda_1} \sum_{\mu \subset \lambda^*} (-1)^{p-\lambda_1} 2^{n(p,\mu)} \sigma_p \sigma_\mu.$$

This result is proved in [10].

2 Quantum Giambelli for $\text{IG}(n - k, 2n)$

The quantum cohomology ring $\text{QH}(\text{IG})$ is a $\mathbb{Z}[q]$ -algebra which is isomorphic to $H^*(\text{IG}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of the formal variable q here is $n + k + 1$. We begin by recalling the quantum Pieri rule of [5]. This states that for any k -strict partition $\lambda \in \mathcal{P}(k, n)$ and integer $p \in [1, n + k]$, we have

$$\sigma_p \cdot \sigma_\lambda = \sum_{\lambda \rightarrow \mu} 2^{N(\lambda,\mu)} \sigma_\mu + \sum_{\lambda \rightarrow \nu} 2^{N(\lambda,\nu)-1} \sigma_{\nu^*} q \quad (11)$$

in the quantum cohomology ring of $\text{IG}(n - k, 2n)$. The first sum in (11) is over partitions $\mu \in \mathcal{P}(k, n)$ such that $|\mu| = |\lambda| + p$, and the second sum is over partitions $\nu \in \mathcal{P}(k, n + 1)$ with $|\nu| = |\lambda| + p$ and $\nu_1 = n + k + 1$.

Proposition 4 *There exists a unique ring homomorphism*

$$\pi : \mathbb{H}(\text{IG}_k) \rightarrow \text{QH}(\text{IG}(n - k, 2n)) \otimes \mathbb{Q}$$

such that the following relations are satisfied:

$$\pi(\sigma_i) = \begin{cases} \sigma_i & \text{if } 1 \leq i \leq n+k, \\ q/2 & \text{if } i = n+k+1, \\ 0 & \text{if } n+k+1 < i \leq 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k. \end{cases}$$

Furthermore, we have $\pi(\sigma_\lambda) = \sigma_\lambda$ for each $\lambda \in \mathcal{P}(k, n)$.

Proof Recall that $\mathbb{H}(\text{IG}_k)$ is the polynomial ring generated by all classes σ_i for $i \geq 1$, modulo the relations (3). These relations for $r > n+k$ uniquely specify the values $\pi(\sigma_i)$ for even integers $i > 2n+2k$. The quantum Pieri rule implies that the remaining relations (3) for $k < r \leq n+k$ are preserved by π .

We next prove that $\pi(\sigma_\lambda) = \sigma_\lambda$ for each $\lambda \in \mathcal{P}(k, n)$. This is clear when λ has only one part. When λ has more than one part, we apply the ring homomorphism π to both sides of (8) and use induction on $\ell(\lambda)$ to show that

$$\pi(\sigma_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p,\mu} \sigma_p \sigma_\mu + \frac{q}{2} \sum_{\mu \subset \lambda^*} a_{n+k+1,\mu} \sigma_\mu \quad (12)$$

holds in $\text{QH}(\text{IG}(n-k, 2n)) \otimes \mathbb{Q}$. We also deduce from Proposition 3 that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p,\mu} \sigma_p \sigma_\mu + \sum_{\mu \subset \lambda^*} a_{n+k+1,\mu} \sigma_{(n+k+1,\mu)} \quad (13)$$

holds in the cohomology ring of $\text{IG}(n+1-k, 2n+2)$. The quantum Pieri rule and (13) imply that the right hand side of (12) evaluates to σ_λ , as desired. \square

Theorem 1 (Quantum Giambelli for IG) *For every $\lambda \in \mathcal{P}(k, n)$, the quantum Giambelli formula for σ_λ in $\text{QH}(\text{IG}(n-k, 2n))$ is obtained from the classical Giambelli formula $\sigma_\lambda = R^\lambda m_\lambda$ in $H^*(\text{IG}(n+1-k, 2n+2), \mathbb{Z})$ by replacing the special Schubert class σ_{n+k+1} with $q/2$.*

Proof This follows from Proposition 4 and Corollary 1. \square

3 Quantum Giambelli for $\text{OG}(n-k, 2n+1)$

3.1 Classical Giambelli for OG

For each $k \geq 0$, let $\text{OG} = \text{OG}(n-k, 2n+1)$ denote the odd orthogonal Grassmannian which parametrizes the $(n-k)$ -dimensional isotropic subspaces in \mathbb{C}^{2n+1} , equipped with a non-degenerate symmetric bilinear form. The Schubert varieties in OG are

indexed by the same set of k -strict partitions $\mathcal{P}(k, n)$ as for $\text{IG}(n - k, 2n)$. Given any $\lambda \in \mathcal{P}(k, n)$ and a complete flag of subspaces

$$F_{\bullet} : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n+1} = \mathbb{C}^{2n+1}$$

such that $F_{n+i} = F_{n+1-i}^{\perp}$ for $1 \leq i \leq n + 1$, we define the codimension $|\lambda|$ Schubert variety

$$X_{\lambda}(F_{\bullet}) = \{\Sigma \in \text{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j(\lambda)}) \geq j \quad \forall 1 \leq j \leq \ell(\lambda)\},$$

where

$$\overline{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}.$$

Let $\tau_{\lambda} \in H^{2|\lambda|}(\text{OG}, \mathbb{Z})$ denote the cohomology class dual to the cycle given by $X_{\lambda}(F_{\bullet})$.

Let $\ell_k(\lambda)$ be the number of parts λ_i which are strictly greater than k , and let \mathcal{Q}_{IG} and \mathcal{Q}_{OG} denote the universal quotient vector bundles over $\text{IG}(n - k, 2n)$ and $\text{OG}(n - k, 2n + 1)$, respectively. It is known (see e.g. [1, Sect. 3.1]) that the map which sends $\sigma_p = c_p(\mathcal{Q}_{\text{IG}})$ to $c_p(\mathcal{Q}_{\text{OG}})$ for all p extends to a ring isomorphism $\varphi : H^*(\text{IG}, \mathbb{Q}) \rightarrow H^*(\text{OG}, \mathbb{Q})$ such that $\varphi(\sigma_{\lambda}) = 2^{\ell_k(\lambda)} \tau_{\lambda}$ for all $\lambda \in \mathcal{P}(k, n)$.

We let $c_p = c_p(\mathcal{Q}_{\text{OG}})$. The *special Schubert classes* on OG are related to the Chern classes c_p by the equations

$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

For any integer sequence α , set $m_{\alpha} = \prod_i c_{\alpha_i}$. Then for every $\lambda \in \mathcal{P}(k, n)$, the classical Giambelli formula

$$\tau_{\lambda} = 2^{-\ell_k(\lambda)} R^{\lambda} m_{\lambda}$$

holds in $H^*(\text{OG}, \mathbb{Z})$.

3.2 From classical to quantum Giambelli

Suppose $k \geq 1$. The quantum cohomology ring $\text{QH}(\text{OG}(n - k, 2n + 1))$ is defined similarly to that of IG, but the degree of q here is $n + k$. More notation is required to state the quantum Pieri rule for OG. For each λ and μ with $\lambda \rightarrow \mu$, we define $N'(\lambda, \mu)$ to be equal to the number (respectively, one less than the number) of connected components of \mathbb{A} , if $p \leq k$ (respectively, if $p > k$). Let $\mathcal{P}'(k, n + 1)$ be the set of $\nu \in \mathcal{P}(k, n + 1)$ for which $\ell(\nu) = n + 1 - k$, $2k \leq \nu_1 \leq n + k$, and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 1$. For any $\nu \in \mathcal{P}'(k, n + 1)$, we let $\tilde{\nu} \in \mathcal{P}(k, n)$ be the partition obtained by removing the first row of ν as well as $n + k - \nu_1$ boxes from the first column. That is,

$$\tilde{v} = (v_2, v_3, \dots, v_r), \text{ where } r = v_1 - 2k + 1.$$

According to [5, Thm. 2.4], for any k -strict partition $\lambda \in \mathcal{P}(k, n)$ and integer $p \in [1, n + k]$, the following quantum Pieri rule holds in $\text{QH}(\text{OG}(n - k, 2n + 1))$.

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda \rightarrow \mu} 2^{N'(\lambda, \mu)} \tau_\mu + \sum_{\lambda \rightarrow v} 2^{N'(\lambda, v)} \tau_{\tilde{v}} q + \sum_{\lambda^* \rightarrow \rho} 2^{N'(\lambda^*, \rho)} \tau_{\rho^*} q^2. \quad (14)$$

Here the first sum is classical, the second sum is over $v \in \mathcal{P}'(k, n + 1)$ with $\lambda \rightarrow v$ and $|v| = |\lambda| + p$, and the third sum is empty unless $\lambda_1 = n + k$, and over $\rho \in \mathcal{P}(k, n)$ such that $\rho_1 = n + k$, $\lambda^* \rightarrow \rho$, and $|\rho| = |\lambda| - n - k + p$.

Let $\delta_p = 1$, if $p \leq k$, and $\delta_p = 2$, otherwise. The stable cohomology ring $\mathbb{H}(\text{OG}_k)$ has a free \mathbb{Z} -basis of Schubert classes τ_λ for k -strict partitions λ , and is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \tau_2, \dots]$ modulo the relations

$$\tau_r^2 + 2 \sum_{i=1}^r (-1)^i \delta_{r-i} \tau_{r+i} \tau_{r-i} = 0 \quad \text{for } r > k. \quad (15)$$

Proposition 5 *There exists a unique ring homomorphism*

$$\tilde{\pi} : \mathbb{H}(\text{OG}_k) \rightarrow \text{QH}(\text{OG}(n - k, 2n + 1))$$

such that the following relations are satisfied:

$$\tilde{\pi}(\tau_i) = \begin{cases} \tau_i & \text{if } 1 \leq i \leq n + k, \\ 0 & \text{if } n + k < i < 2n + 2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n + 2k. \end{cases}$$

Furthermore, we have $\tilde{\pi}(\tau_\lambda) = \tau_\lambda$ for each $\lambda \in \mathcal{P}(k, n)$.

Proof The relations (15) for $r \geq n + k$ uniquely specify the values $\tilde{\pi}(\tau_i)$ for even integers $i \geq 2n + 2k$. We must show that the remaining relations for $k < r < n + k$ are mapped to zero by $\tilde{\pi}$. Observe that when $k < n - 1$ the individual terms in these relations carry no q correction. Indeed, we are applying the quantum Pieri rule (14) to partitions of length one, hence the q term vanishes (since $1 < n - k$) and the q^2 term vanishes (since $\deg(q^2) = 2n + 2k$). It remains only to consider the case $k = n - 1$, which uses the quantum Pieri rule for the quadric $\text{OG}(1, 2n + 1)$. The computation is then done as in [5, Thm. 2.5] (which treats the case $r = n$), and involves computing the coefficient c of $q \tau_{2(r-n)+1}$ in the corresponding expression. As in loc. cit., the result is $c = 1 - 2 + 2 - \dots \pm 2 \mp 1$ when $r \leq (3n - 2)/2$, and otherwise $c = 2 - 4 + 4 - \dots \pm 4 \mp 2$; hence $c = 0$ in both cases.

To prove that $\tilde{\pi}(\tau_\lambda) = \tau_\lambda$ for every $\lambda \in \mathcal{P}(k, n)$, we use an orthogonal analogue of Proposition 3, which follows from the isomorphism $\mathbb{H}(\text{OG}_k) \otimes \mathbb{Q} \cong \mathbb{H}(\text{IG}_k) \otimes \mathbb{Q}$. Arguing by induction on $\ell(\lambda)$ as in Proposition 4, we obtain that

$$\tilde{\pi}(\tau_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a'_{p,\mu} \tau_p \tau_\mu \quad (16)$$

holds in $\mathrm{QH}(\mathrm{OG}(n-k, 2n+1)) \otimes \mathbb{Q}$, where $a'_{p,\mu} \in \mathbb{Q}$. The quantum Pieri rule (14) implies that any product $\tau_p \tau_\mu$ in (16) carries no q correction terms. It follows that the right hand side of (16) evaluates to τ_λ . \square

Theorem 2 (Quantum Giambelli for OG) *For every $\lambda \in \mathcal{P}(k, n)$, we have*

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda m_\lambda$$

in the quantum cohomology ring $\mathrm{QH}(\mathrm{OG}(n-k, 2n+1))$. In other words, the quantum Giambelli formula for OG is the same as the classical Giambelli formula.

Proof This follows from Proposition 5 and the orthogonal version of Corollary 1. \square

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